

Knotted domain strings

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We construct meta-stable knotted domain strings on the surface of a soliton of the shape of a torus in 3+1 dimensions. We consider the simplest case of \mathbb{Z}_2 Wess-Zumino-type domain walls for which we can cover the torus with a domain string accompanied with an anti-domain string. In this theory, all (p, q) -torus knots can be realized as a linked pair of a(n) (un)knotted domain string and an anti-domain string.

Knotted vortex strings — More than 140 years ago, Lord Kelvin proposed an interesting idea that atoms could be conceived as stable knotted vortex loops. Although this idea was not successful as a theory of atoms, it led to the celebrated mathematical knot theory today. Knots are one of the most fascinating structures frequently appearing in Nature and they are found to be important in diverse areas of physics such as high energy physics, cosmology and condensed matter physics.

It was a long-standing question whether a stable knotted structure actually exists in a dynamical system. Indeed, until quite recently, no stable knot structures were found. In 1996, Gladikowski and Hellmund [1] as well as Faddeev and Niemi [2] found stable knot-like structures made of (stable) topological solitons, Hopfions, which are (un)knotted closed loops of vortex tubes in the Faddeev-Skyrme model [3]. The Faddeev-Skyrme model is an $O(3)$ nonlinear sigma model with the addition of a (four-derivative) Skyrme term. With the aid of the recent drastically improved computer power, they succeeded in constructing numerical solutions of Hopfions with small charges. Faddeev and Niemi [2] conjectured that all torus knots can be constructed from stable knotted vortex tubes. Soon after, Battye and Sutcliffe found beautiful higher-charged Hopfions numerically, which have both link and knot structures [4].

After the discovery in terms of numerical solutions, the knotted solitons have been studied extensively in the literature. However, all knotted solitons, known so far, are obtained from closed vortex flux tubes. The purpose of this Letter is to demonstrate the existence of a different type of knotted soliton, which lives on the surface of toroidal structures in 3+1 dimensions. Viz. we construct non-planar domain strings on the surface of a ring-shaped soliton, like a vorton [5], a superconducting string loop (spring) [6] or a Q -ring [7] etc.

The model — In [8, 9] it was proposed that non-planar domain wall networks might exist on the surface of a host soliton star which is a spherically compactified domain wall. This idea was investigated numerically in [10] in a simple concrete model having two complex scalar fields and $U(1) \times \mathbb{Z}_n$ global symmetry. In [10], the host soliton (star) is a large Q -ball [11] and an attempt was made to tile the surface of the Q -ball with almost-BPS planar

domain-string networks in the Wess-Zumino model [12]. However, in [10] it was proven that only a few spherical polyhedra can be constructed on the surface of a sphere.

Although an attempt of tiling the domain-string network on a sphere did not turn out very successful, we can still ask whether it is possible to tile other Riemann surfaces with domain strings. Namely, how does the answer depend on the topology of the host soliton. Because we are interested in a solitonic object which may be dynamically produced, we choose a torus, T^2 , which is one of the simplest geometries. Indeed, many ring-shaped solitons are known, e.g. a closed loop of the superconducting cosmic strings [5], springs [6] and Hopfions [1].

We will work in the framework of the simple model proposed in [10] which was used to construct the domain-string networks on Q -balls. The Lagrangian contains two complex scalar fields ψ and ϕ

$$\mathcal{L} = \mathcal{L}_1[\psi] + \mathcal{L}_2[\psi, \phi]. \quad (1)$$

For $\mathcal{L}_1[\psi]$, we can choose any model admitting a ring-shaped soliton. Instead of specifying the details of the ring soliton, we only need to assume that the profile function of ψ obeys $|\psi| = 1$ inside and $\psi = 0$ outside of the ring, see Fig. 1. Then, for $\mathcal{L}_2[\psi, \phi]$, we consider the spe-

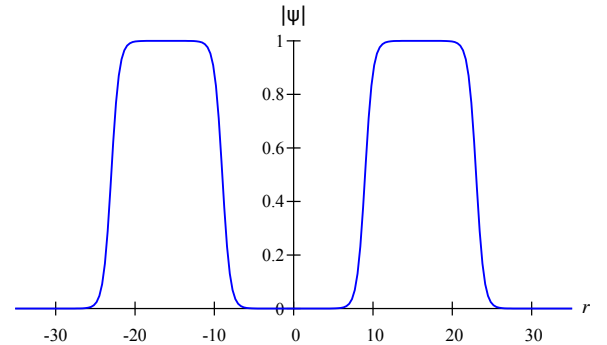


FIG. 1: The cross-section profile of ψ for a ring-shaped soliton at $z = 0$. The ring is axially symmetric with respect to the z -axis.

cific model

$$\mathcal{L}_2 = \frac{1}{2} |\partial_\mu \phi|^2 - \beta^2 \left| \eta[\psi] - \frac{\phi^2}{v^2} \right|^2, \quad (2)$$

$$\eta[\psi] \equiv 4|\psi|^2 (1 - |\psi|^2). \quad (3)$$

First we note that $\phi = 0$ is a solution of the equations of motion. As done in [10], we then treat β as a small parameter (for concreteness we take $\beta = 0.1$), such that the profile of the torus configuration of \mathcal{L}_1 receives only a negligible correction which can thus be ignored. Since $\eta = 0$ both inside and outside of the ring, ϕ develops a non-zero VEV only on the surface of the ring. Therefore, \mathcal{L}_2 is effectively the 2+1 dimensional Wess-Zumino model with a \mathbb{Z}_2 discrete symmetry, where the field ϕ has VEVs $\langle \phi \rangle \simeq \pm v$. As well known, the Wess-Zumino model in 2+1 dimensions admits domain strings interpolating those two vacua having tension and transverse size, respectively,

$$T = \frac{2^{5/2}}{3} \beta v, \quad d \simeq \frac{v}{\beta}. \quad (4)$$

In the following, we will tile the surface of the ring soliton with these two domains.

Knotted domain strings — As we will explain, domain strings on the ring surface are nothing but torus knots. Therefore, they are naturally characterized by a pair of co-prime integers (p, q) . The number p denotes the poloidal winding number (around the meridian circle), while q is the toroidal winding number (around the longitudinal circle) of the torus. As well known, torus knots are prime and chiral. Torus knots with $pq > 0$ are right-handed and $pq < 0$ are left-handed. A (p, q) -knot is identical to the $(-p, -q)$ -knot.

The simplest configurations are the $(1, 0)$ and $(0, 1)$. These have unknotted closed domain strings sitting on the antipodal points of the torus, see Fig. 2. Clearly, the configuration $(0, 1)$ is unstable against small perturbations because the smaller string loop is preferred energetically. Thus, the larger string will shrink and annihilate the smaller one since the net \mathbb{Z}_2 charge of the configuration is trivial.

The Hopf link appears for the $(1, 1)$ type. The configuration is unknotted but two domain strings are singly linked which is a so-called Hopf link. Each string loop winds both cycles of the torus once. The domain string and anti-domain string always sit on the antipodal point with respect to the other on the torus. They are thus maximally separated from each other.

The unknotted but doubly linked strings are obtained for the $(2, 1)$ and $(1, 2)$ cases. They are called Solomon's links, see Fig. 4. Although the $(2, 1)$ and $(1, 2)$ are identical in a topological sense, their energy density distributions are distinct. Hence, we can physically distinguish those two configurations.

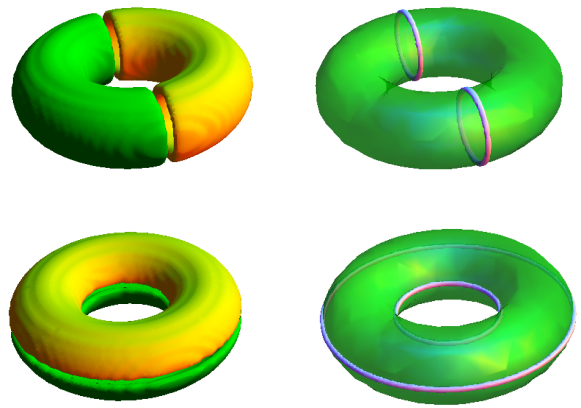


FIG. 2: The left panel shows the two domains. The two domain strings (i.e. energy density isosurface) are shown in the right panel. The upper figures are for the $(1, 0)$ type, i.e. poloidal strings, while the lower figures are for the $(0, 1)$ type, i.e. toroidal strings.

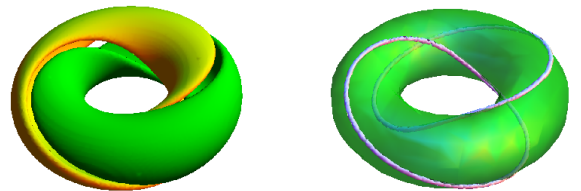


FIG. 3: Hopf link $(1, 1)$: The left figure shows the tiling the surface of a torus with two domains and the right figure shows that the domain strings are linked once.

The next example is the linked trefoil which is the simplest knotted structure among the torus knots. Namely, they are characterized by winding numbers $(3, 2)$ and $(2, 3)$, see Figs. 5 and 6. The linking number is the product of p and q . Hence, both these trefoils are linked 6 times. In [2], it was conjectured that all torus knots can be constructed as Hopfions in the Faddeev-Skyrme model, but to the best of our knowledge our solutions are the first ones realizing both the $(3, 2)$ and the $(2, 3)$ configuration.

The last example is the linked knot with winding numbers $(4, 3)$ and linking number 12, see Fig. 7.

Stability — Some comments on the stability of the domain strings are in order. Since \mathcal{L}_2 is effectively the \mathbb{Z}_2 Wess-Zumino model on the surface of a torus, the domain strings are characterized by two winding numbers, (p, q) , and also by the \mathbb{Z}_2 charge. Each string has either $+$ or $-$ \mathbb{Z}_2 charge. All configurations are topologically protected in the sense of the topologically non-trivial winding numbers (p, q) . However, since the torus surface is

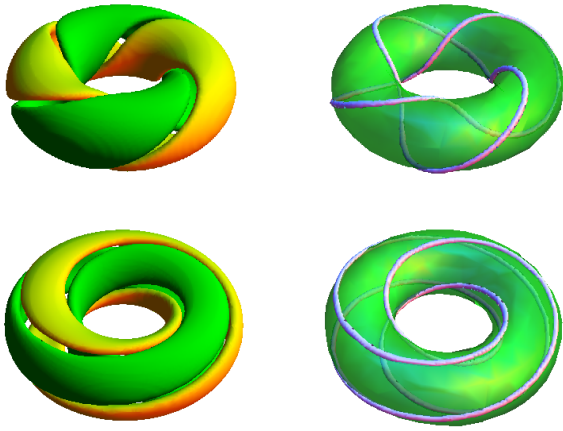


FIG. 4: Solomon's links: The upper figures show the $(2,1)$ configuration and the lower ones show the $(1,2)$ -type strings. These are unknotted but doubly linked configurations.

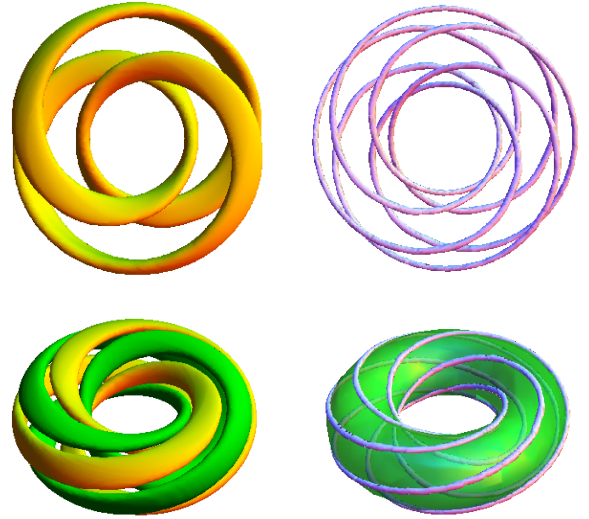


FIG. 6: The sixthly linked trefoil knot with winding numbers $(2,3)$. See the caption of Fig. 5.

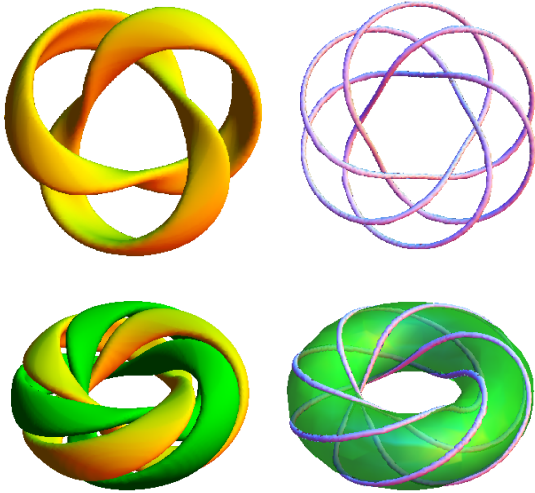


FIG. 5: The sixthly linked trefoil knot with winding numbers $(3,2)$. The top-left figure shows the region where $\text{Re}[\phi] > 0$ which is a trefoil form. The top-right figure displays the two linked trefoil domain strings in the view from above. The bottom-left figure shows the two domains on the torus surface and the bottom-right figure is a 3D view of the trefoil knots.

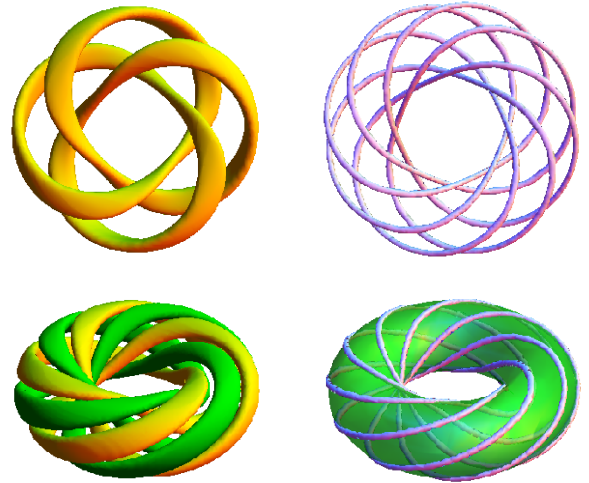


FIG. 7: The linked knot with winding numbers $(4,3)$.

periodic, a single domain string is always accompanied by its anti-domain string. Hence, the total \mathbb{Z}_2 charge is trivial. Therefore, all the domain-string configurations which we have obtained in this letter are non-topological solitons.

Although their stability is granted by topology, we expect meta-stability and thus a sufficiently long life time of the configurations. This is due to the interaction between the \mathbb{Z}_2 kinks being exponentially suppressed as $V \sim \exp(-mL)$ where m is a mass parameter which is

of the order of the inverse kink width d and L denotes the separation distance of the \mathbb{Z}_2 kink and anti-kink. Indeed, the asymptotic potential can be numerically calculated with the superposition of the well-known kink and anti-kink solutions, and one finds that the potential has a large plateau in the asymptotic region. Only when the two kinks are close enough together, they feel a finite (non-negligible) attractive force. Since the two domain strings in all the constructed configurations are maximally separated, the attractive interactions among them are small enough to render the configurations sufficiently stable. An exception, however, is the unknot of type $(0,1)$ which clearly is unstable against small pertur-

bations.

When we increase either winding number, p or q , a fixed size torus leaves less and less space for the individual domain string in order not to be too close to its anti-domain string. That is, when that happens, they will simply annihilate and leave behind a single vacuum. Hence, a crude estimate of the maximum allowed winding numbers of the strings are

$$d \ll L \sim \frac{\pi R_P R_T}{\sqrt{q^2 R_T^2 + p^2 R_P^2}}, \quad (5)$$

where R_P is the poloidal radius, R_T is the inner toroidal radius. Here, we took $d \simeq v/\beta = 1$ and $R_{P,T} = \mathcal{O}(10)$.

A further point in favor of our argument of stability comes from the numerical method of solving the static equations of motion which we utilized. Viz. in order to obtain static solutions, we solve the variational problem of the energy of the model \mathcal{L}_2 . To this end, we solve the associated so-called gradient flow equations: $\partial_\tau \phi = -\partial_\phi \mathcal{E}$ with τ being relaxation time [4]. We solved these differential equations in two different ways; i.e. in *Mathematica* on a 80^3 lattice with a lattice spacing of 0.875 and also using a Crank-Nicolson algorithm in conjunction with a biconjugate gradients method and iterating the linearized equation to obtain a solution to the nonlinear equation of motion. With an appropriate initial configuration, one obtains the desired results as final states of the relaxation. If the configuration is unstable, it collapses to a single vacuum. Since we obtained non-trivial domain strings by means of the relaxation, they represent stationary points of the energy.

There are three options for improving the stability. i) Charging the domain strings in such away that they repel or attract each other by some kind of confining force. ii) A stable bound state of a string and anti-string may exist. iii) Considering a periodic model as \mathcal{L}_2 . All of these may be realized by changing the model \mathcal{L}_2 . The third option might be the best choice. As an example, we can choose a modified sine-Gordon model for \mathcal{L}_2 with a periodic field à la axion field. Because the sine-Gordon model is periodic by definition, we do not need the anti-domain string if we stick in a branch cut circle on the torus. Thus, the single domain string can exist by itself in such a model. We will report on this possibility elsewhere.

Conclusions — In this Letter, we obtained numerical solutions of new knotted domain strings on the surface of a ring soliton in $3 + 1$ dimensions. We found several torus knots with winding numbers $(1, 0)$, $(2, 1)$, $(3, 2)$ and $(4, 3)$. With these results, we expect that all torus knots, i.e. with any co-prime integers (p, q) can be constructed as domain strings in dynamical systems [13]. It is known that lots of complicated three-dimensional shapes can be

formed as solitons, for instance, the Buckyball was found as a higher-charged Skyrmion. Even on such a complicated two-dimensional surface as host soliton, we may construct domain strings. We would like to emphasize that this is indeed a doable task since the method of [10] is really simple and changing the topology of the host soliton does not lead to any difficulties. We hope that the knotted domain strings found here will open new research directions in many areas of physics and mathematics.

Acknowledgments — The work of M. E. is supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan (No. 23740226) and Japan Society for the Promotion of Science (JSPS) and Academy of Sciences of the Czech Republic (ASCR) under the Japan - Czech Republic Research Cooperative Program. The work of S. B. G. is partially supported by the American-Israeli Bi-National Science Foundation and the Israel Science Foundation Center of Excellence.

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